

# Integral Equation for Scattering of Light by a Strong Magnetostatic Field in Vacuum

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## Abstract

When a strong magnetostatic field is present, vacuum effectively appears as a linear, uniaxial, dielectric–magnetic medium for small–magnitude optical fields. The availability of the frequency–domain dyadic Green function when the magnetostatic field is spatially uniform facilitates the formulation of an integral equation for the scattering of an optical field by a spatially varying magnetostatic field in vacuum. This integral equation can be numerically treated by using the method of moments as well as the coupled dipole method. Furthermore, the principle underlying the strong–property–fluctuation theory allows the homogenization of a spatially varying magnetostatic field in the context of light scattering.

*Key words:* coupled dipole method, depolarization, homogenization, method of moments, quantum electrodynamics,

# 1 Introduction

In classical electrodynamics, light propagating in vacuum (i.e., matter-free space) is not considered to be affected by the presence of a magnetostatic field. This is because classical vacuum is a linear medium wherein the principle of superposition holds. But, in quantum electrodynamics (QED), vacuum is a nonlinear medium (Jackson, 1998). It can, however, be linearized for a rapidly time-varying electromagnetic field with a small amplitude in the presence of a slowly varying (or static) magnetic field (Adler, 2007). The price of linearization is that the QED vacuum appears as an anisotropic dielectric-magnetic medium for optical fields (Adler, 1971).

Our modest aim in this paper is to derive an integral equation for the scattering of a high-frequency electromagnetic field (typified by light) by a strong magnetostatic field in vacuum. For this purpose, we exploit the analytical machinery developed during the last two decades for the frequency-domain analysis of electromagnetic fields in complex mediums (Singh & Lakhtakia, 2000; Weiglhofer & Lakhtakia, 2003). As the derived integral equation shall have to be solved numerically in general, we also outline the solution strategies provided by the method of moments (Miller, Medgyesi-Mitschang, & Newman, 1991; Wang, 1991) and the coupled dipole method (Purcell & Pennypacker, 1973; Lakhtakia, 1990). Finally, we adopt the principle underlying the strong-property-fluctuation theory to homogenize the magnetostatic field in the context of light scattering.

A note about notation: 3-vectors (6-vectors) are in normal (bold) face and underlined, whereas  $3 \times 3$  dyadics ( $6 \times 6$  dyadics) are in normal (bold) face and double underlined. The position vector is denoted by  $\underline{r} = x\underline{\hat{u}}_x + y\underline{\hat{u}}_y + z\underline{\hat{u}}_z$  in a Cartesian coordinate system with unit vectors  $\underline{\hat{u}}_x$ ,  $\underline{\hat{u}}_y$ , and  $\underline{\hat{u}}_z$ , whereas time is denoted by  $t$ . The real part of a complex-valued quantity  $\zeta$  is written as  $\Re\{\zeta\}$ .

## 2 Constitutive Equations for Optical Fields

Suppose that all space is matter-free and that a magnetostatic field  $\underline{B}^{dc}(\underline{r})$  is present everywhere. Optical fields (superscripted “o” in this paper) satisfy the source-free Maxwell equations (Adler,

1971)

$$\left. \begin{aligned} \nabla \cdot \underline{B}^o(\underline{r}, t) &= 0 \\ \nabla \cdot \underline{D}^o(\underline{r}, t) &= 0 \\ \nabla \times \underline{E}^o(\underline{r}, t) &= -\frac{\partial}{\partial t} \underline{B}^o(\underline{r}, t) \\ \nabla \times \underline{H}^o(\underline{r}, t) &= \frac{\partial}{\partial t} \underline{D}^o(\underline{r}, t) \end{aligned} \right\}, \quad (1)$$

provided  $|\underline{B}^o| \ll |\underline{B}^{dc}(\underline{r})| \forall \underline{r}$ . As per the linearized version of QED, the optical fields appearing in the foregoing equations obey the constitutive equations (Adler,1971)

$$\left. \begin{aligned} \underline{D}^o(\underline{r}, t) &= \epsilon_0 \underline{\underline{\epsilon}}^o(\underline{r}) \cdot \underline{E}^o(\underline{r}, t) \\ \underline{H}^o(\underline{r}, t) &= \mu_0^{-1} \underline{\underline{\nu}}^o(\underline{r}) \cdot \underline{B}^o(\underline{r}, t) \end{aligned} \right\}, \quad (2)$$

where  $\epsilon_0 = 8.8542 \times 10^{-12} \text{ F m}^{-1}$  and  $\mu_0 = 4\pi \times 10^{-7} \text{ H m}^{-1}$  are constants characterizing the classical vacuum, whereas the relative permittivity dyadic

$$\underline{\underline{\epsilon}}^o(\underline{r}) = \left[ 1 - 8\epsilon_0 c_0^2 \xi \underline{B}^{dc}(\underline{r}) \cdot \underline{B}^{dc}(\underline{r}) \right] \underline{I} + 28\epsilon_0 c_0^2 \xi \underline{B}^{dc}(\underline{r}) \underline{B}^{dc}(\underline{r}) \quad (3)$$

and the relative impermeability dyadic<sup>1</sup>

$$\underline{\underline{\nu}}^o(\underline{r}) = \left[ 1 - 8\epsilon_0 c_0^2 \xi \underline{B}^{dc}(\underline{r}) \cdot \underline{B}^{dc}(\underline{r}) \right] \underline{I} - 16\epsilon_0 c_0^2 \xi \underline{B}^{dc}(\underline{r}) \underline{B}^{dc}(\underline{r}) \quad (4)$$

emerge from QED to dictate the influence of the magnetostatic field of high magnitude on the optical field. Here,  $c_0 = 1/\sqrt{\epsilon_0 \mu_0}$  is the speed of light in classical vacuum, whereas

$$\xi = \frac{(e_{e\ell}/m_{e\ell})^4 \hbar}{45(4\pi\epsilon_0)^2 c_0^7} = 8.3229 \times 10^{-32} \text{ kg}^{-1} \text{ m s}^2 \quad (5)$$

contains the electronic charge  $e_{e\ell} = 1.6022 \times 10^{-19} \text{ C}$ , the electronic mass  $m_{e\ell} = 9.1096 \times 10^{-31} \text{ kg}$ , and the reduced Planck constant  $\hbar = 1.0546 \times 10^{-34} \text{ J s}$ . Clearly, the QED vacuum appears to the optical field as a spatiotemporally local, spatially nonhomogeneous, temporally unvarying, uniaxial dielectric–magnetic medium. Set  $\hbar = 0$  to convert from the QED vacuum to the classical vacuum, and the influence of the magnetostatic field on the optical field vanishes.

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<sup>1</sup>Impermeability is the reciprocal of the permeability.

### 3 Dyadic Green Function for Uniform Magnetostatic Field

Suppose that  $\underline{B}^{dc}(\underline{r}) = \underline{B}_c$  is spatially uniform everywhere, so that the QED vacuum then is spatially homogeneous. Defining

$$\left. \begin{aligned} a_c &= \epsilon_0 c_0^2 \xi |\underline{B}_c|^2 \\ \underline{u}_c &= \underline{B}_c / |\underline{B}_c| \end{aligned} \right\}, \quad (6)$$

we can write the relative permittivity dyadic

$$\underline{\underline{\epsilon}}_c^o = (1 - 8a_c)(\underline{\underline{I}} - \underline{u}_c \underline{u}_c) + (1 + 20a_c) \underline{u}_c \underline{u}_c \quad (7)$$

and the relative permeability dyadic

$$\underline{\underline{\mu}}_c^o = \left( \underline{\underline{\nu}}_c^o \right)^{-1} = \frac{1}{1 - 8a_c} (\underline{\underline{I}} - \underline{u}_c \underline{u}_c) + \frac{1}{1 - 24a_c} \underline{u}_c \underline{u}_c \quad (8)$$

for the QED vacuum, with  $\underline{\underline{I}}$  as the 3×3 identity dyadic. The propagation of optical plane waves, characterized by

$$\underline{E}^o(\underline{r}, t) = \Re \left\{ \underline{\tilde{E}}^o \exp[i(\underline{k} \cdot \underline{r} - \omega t)] \right\}, \quad (9)$$

etc., with amplitude vector  $\underline{\tilde{E}}^o$ , wave vector  $\underline{k}$ , and angular frequency  $\omega$  was analyzed by Adler (1971) using standard mathematical techniques. Both the relative permittivity and the relative permeability dyadics are uniaxial and share the same distinguished axis. Therefore, the medium is optically birefringent except when  $\underline{k}$  is parallel to  $\underline{u}_c$ ; furthermore, both plane waves propagating in a fixed direction are to be classified as *extraordinary* in optical parlance (Lakhtakia, Varadan, & Varadan, 1991).

For a more general consideration of frequency-domain optical fields, it is best to write

$$\underline{E}^o(\underline{r}, t) = \Re \{ \underline{\epsilon}^o(\underline{r}) \exp(-i\omega t) \}, \quad (10)$$

etc., where  $\underline{\epsilon}^o(\underline{r})$  is a phasor. The Maxwell curl equations may then be compactly recast with the help of 6-vectors and 6×6 dyadics as

$$\left[ \underline{\underline{L}}(\nabla) + i\omega \underline{\underline{C}}_c^o \right] \cdot \underline{\mathbf{f}}^o(\underline{r}) = \underline{\mathbf{s}}^o(\underline{r}), \quad (11)$$

where

$$\underline{\underline{L}}(\nabla) = \begin{bmatrix} \underline{\underline{0}} & \nabla \times \underline{\underline{I}} \\ -\nabla \times \underline{\underline{I}} & \underline{\underline{0}} \end{bmatrix}, \quad (12)$$

$$\underline{\underline{\mathbf{C}}}_c^o = \begin{bmatrix} \epsilon_0 \underline{\underline{\epsilon}}_c^o & \underline{\underline{0}} \\ \underline{\underline{0}} & \mu_0 \underline{\underline{\mu}}_c^o \end{bmatrix}, \quad (13)$$

$$\underline{\mathbf{f}}^o(\underline{\mathbf{r}}) = \begin{bmatrix} \underline{\mathbf{e}}^o(\underline{\mathbf{r}}) \\ \underline{\mathbf{h}}^o(\underline{\mathbf{r}}) \end{bmatrix}, \quad (14)$$

$\underline{\mathbf{s}}^o(\underline{\mathbf{r}})$  represents the (high-frequency) electric and magnetic source current densities that engender  $\underline{\mathbf{f}}^o(\underline{\mathbf{r}})$ , and  $\underline{\underline{0}}$  is the  $3 \times 3$  null dyadic.

Equation (11) is converted into the integral equation

$$\underline{\mathbf{f}}^o(\underline{\mathbf{r}}) = \underline{\mathbf{f}}_h^o(\underline{\mathbf{r}}) + \int d^3 \underline{\mathbf{r}}' \underline{\underline{\mathbf{g}}}_c^o(\underline{\mathbf{r}}, \underline{\mathbf{r}}') \cdot \underline{\mathbf{s}}^o(\underline{\mathbf{r}}'), \quad (15)$$

where  $\underline{\mathbf{f}}_h^o(\underline{\mathbf{r}})$  is the solution of (11) when the source term on its right side is null-valued *everywhere*.

The  $6 \times 6$  dyadic Green function  $\underline{\underline{\mathbf{g}}}_c^o(\underline{\mathbf{r}} - \underline{\mathbf{r}}')$  is the solution of

$$\left[ \underline{\underline{\mathbf{I}}}(\nabla) + i\omega \underline{\underline{\mathbf{C}}}_c^o \right] \cdot \underline{\underline{\mathbf{g}}}_c^o(\underline{\mathbf{r}}, \underline{\mathbf{r}}') = \underline{\underline{\mathbf{I}}} \delta(\underline{\mathbf{r}} - \underline{\mathbf{r}}'), \quad (16)$$

where  $\underline{\underline{\mathbf{I}}}$  is the  $6 \times 6$  identity dyadic and  $\delta(\underline{\mathbf{r}} - \underline{\mathbf{r}}')$  is Dirac delta function.

The  $6 \times 6$  dyadic Green function  $\underline{\underline{\mathbf{g}}}_c^o(\underline{\mathbf{r}}, \underline{\mathbf{r}}')$  is available from the literature on electromagnetic fields in uniaxial mediums as (Weiglhofer, 1990)

$$\underline{\underline{\mathbf{g}}}_c^o(\underline{\mathbf{r}}, \underline{\mathbf{r}}') = \begin{bmatrix} \underline{\underline{\mathbf{I}}} & -(i\omega \epsilon_0 \underline{\underline{\epsilon}}_c^o)^{-1} \cdot (\nabla \times \underline{\underline{\mathbf{I}}}) \\ (i\omega \mu_0 \underline{\underline{\mu}}_c^o)^{-1} \cdot (\nabla \times \underline{\underline{\mathbf{I}}}) & \underline{\underline{\mathbf{I}}} \end{bmatrix} \begin{bmatrix} \underline{\underline{g}}_{ce}^o(\underline{\mathbf{r}}, \underline{\mathbf{r}}') & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{g}}_{cm}^o(\underline{\mathbf{r}}, \underline{\mathbf{r}}') \end{bmatrix}, \quad (17)$$

where the  $3 \times 3$  dyadic Green functions

$$\underline{\underline{g}}_{ce}^o(\underline{\mathbf{r}}, \underline{\mathbf{r}}') = -(1 - 8a_c)^{-1} \left\{ (i\omega \epsilon_0)^{-1} \left[ \nabla \nabla + k_0^2 (1 + 20a_c) (\underline{\underline{\epsilon}}_c^o)^{-1} \right] \gamma_e^o(\underline{\mathbf{r}}, \underline{\mathbf{r}}') + i\omega \mu_0 \underline{\underline{\varphi}}(\underline{\mathbf{r}}, \underline{\mathbf{r}}') \right\} \quad (18)$$

and

$$\underline{\underline{g}}_{cm}^o(\underline{\mathbf{r}}, \underline{\mathbf{r}}') = -(1 - 8a_c) \left\{ (i\omega \mu_0)^{-1} \left[ \nabla \nabla + k_0^2 (1 - 24a_c)^{-1} (\underline{\underline{\mu}}_c^o)^{-1} \right] \gamma_m^o(\underline{\mathbf{r}}, \underline{\mathbf{r}}') - i\omega \epsilon_0 \underline{\underline{\varphi}}(\underline{\mathbf{r}}, \underline{\mathbf{r}}') \right\} \quad (19)$$

employ the usual wavenumber  $k_0 = \omega(\epsilon_0 \mu_0)^{1/2}$  for classical vacuum. The  $3 \times 3$  dyadic function

$$\begin{aligned} \underline{\underline{\varphi}}(\underline{\mathbf{r}}, \underline{\mathbf{r}}') &= \left[ \left( \frac{1 + 20a_c}{1 - 8a_c} \right) \gamma_e^o(\underline{\mathbf{r}}, \underline{\mathbf{r}}') - \left( \frac{1 - 8a_c}{1 - 24a_c} \right) \gamma_m^o(\underline{\mathbf{r}}, \underline{\mathbf{r}}') \right] \frac{(\underline{\mathbf{R}} \times \underline{\mathbf{u}}_c)(\underline{\mathbf{R}} \times \underline{\mathbf{u}}_c)}{|\underline{\mathbf{R}} \times \underline{\mathbf{u}}_c|^2} \\ &+ \left[ \underline{\underline{\mathbf{I}}} - \underline{\underline{\mathbf{u}}_c \underline{\mathbf{u}}_c} - 2 \frac{(\underline{\mathbf{R}} \times \underline{\mathbf{u}}_c)(\underline{\mathbf{R}} \times \underline{\mathbf{u}}_c)}{|\underline{\mathbf{R}} \times \underline{\mathbf{u}}_c|^2} \right] \frac{R_e^o \gamma_e^o(\underline{\mathbf{r}}, \underline{\mathbf{r}}') - R_m^o \gamma_m^o(\underline{\mathbf{r}}, \underline{\mathbf{r}}')}{ik_0 |\underline{\mathbf{R}} \times \underline{\mathbf{u}}_c|^2} \end{aligned} \quad (20)$$

and the scalar functions

$$\gamma_e^o(\underline{\mathbf{r}}, \underline{\mathbf{r}}') = \frac{\exp(ik_0 R_e^o)}{4\pi R_e^o} \quad (21)$$

and

$$\gamma_m^o(\underline{r}, \underline{r}') = \frac{\exp(ik_0 R_m^o)}{4\pi R_m^o} \quad (22)$$

contain

$$\underline{R} = \underline{r} - \underline{r}', \quad (23)$$

$$R_e^o = + \left[ \left( \frac{1+20a_c}{1-8a_c} \right) |\underline{R} \times \underline{u}_c|^2 + (\underline{R} \cdot \underline{u}_c)^2 \right]^{1/2}, \quad (24)$$

$$R_m^o = + \left[ \left( \frac{1-8a_c}{1-24a_c} \right) |\underline{R} \times \underline{u}_c|^2 + (\underline{R} \cdot \underline{u}_c)^2 \right]^{1/2}. \quad (25)$$

## 4 Formulation of Integral Equation

Returning to the more general case of a nonuniform magnetostatic field  $\underline{B}^{dc}(\underline{r})$ , and making use of the representation (10), we see that the two curl equations in (1) may be written together as

$$[\underline{\mathbf{L}}(\nabla) + i\omega \underline{\mathbf{C}}^o(\underline{r})] \cdot \underline{\mathbf{f}}^o(\underline{r}) = \underline{\mathbf{0}}, \quad (26)$$

where  $\underline{\mathbf{0}}$  is the null 6-vector and

$$\underline{\mathbf{C}}^o(\underline{r}) = \begin{bmatrix} \epsilon_0 \underline{\mathbf{E}}^o(\underline{r}) & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \mu_0 [\underline{\mathbf{V}}^o(\underline{r})]^{-1} \end{bmatrix}. \quad (27)$$

As a simple Green-function formalism is unlikely to be found for this more general case, we *define* a suitable uniform magnetostatic field  $\underline{B}_c$  and proceed as follows.

### 4.1 Derivation

Equation (26) is recast as

$$[\underline{\mathbf{L}}(\nabla) + i\omega \underline{\mathbf{C}}_c^o] \cdot \underline{\mathbf{f}}^o(\underline{r}) = \underline{\mathbf{s}}_{eq}^o(\underline{r}), \quad (28)$$

where

$$\underline{\mathbf{s}}_{eq}^o(\underline{r}) = i\omega [\underline{\mathbf{C}}_c^o - \underline{\mathbf{C}}^o(\underline{r})] \cdot \underline{\mathbf{f}}^o(\underline{r}) \quad (29)$$

is an *equivalent* high-frequency source current density 6-vector that contains the spatial variations of  $\underline{B}^{dc}(\underline{r})$  in relation to  $\underline{B}_c$ . Clearly,  $\underline{B}_c$  should be chosen carefully. If the variations of  $\underline{B}^{dc}(\underline{r})$  are confined to some bounded region only,  $\underline{B}_c$  should be chosen as the value of  $\underline{B}^{dc}(\underline{r})$  outside

that region. If  $\underline{B}^{dc}(\underline{r})$  varies over all space,  $\underline{B}_c$  may emerge from some homogenization or spatial-averaging procedure. Once that choice has been made, the solution of (28) — and therefore of (26) — may be written as

$$\underline{\mathbf{f}}^o(\underline{r}) = \underline{\mathbf{f}}_h^o(\underline{r}) + i\omega \int d^3 \underline{r}' \underline{\mathbf{g}}_c^o(\underline{r}, \underline{r}') \cdot \left[ \underline{\mathbf{C}}_c^o - \underline{\mathbf{C}}^o(\underline{r}') \right] \cdot \underline{\mathbf{f}}^o(\underline{r}'), \quad (30)$$

where  $\underline{\mathbf{f}}_h^o(\underline{r})$  is the homogeneous part of the solution of (28).

The integrand on the right side of (30) is singular at  $\underline{r} = \underline{r}'$  and therefore requires additional treatment. An ellipsoidal region  $V_\varepsilon$  containing the location  $\underline{r}$  midway between its two focuses is identified in the domain of integration. The surface of  $V_\varepsilon$  is the set of points

$$\underline{r}_\varepsilon(\theta_q, \phi_q) = \underline{r} + \varepsilon \underline{U} \cdot \hat{\underline{u}}_q, \quad \theta_q \in [0, \pi], \quad \phi_q \in [0, 2\pi], \quad (31)$$

where  $\varepsilon$  is a positive scalar, the unit vector

$$\hat{\underline{u}}_q = (\hat{\underline{u}}_x \cos \phi_q + \hat{\underline{u}}_y \sin \phi_q) \sin \theta_q + \hat{\underline{u}}_z \cos \theta_q, \quad (32)$$

and the dyadic

$$\underline{\underline{U}} = a_x \hat{\underline{u}}_x \hat{\underline{u}}_x + a_y \hat{\underline{u}}_y \hat{\underline{u}}_y + a_z \hat{\underline{u}}_z \hat{\underline{u}}_z \quad (33)$$

has positive eigenvalues  $a_x$ ,  $a_y$ , and  $a_z$ , all less than or equal to unity. The integral is divided into two parts: the first is to be evaluated over  $V_\varepsilon$  only, the second over all space except  $V_\varepsilon$ . Both parts are evaluated in the limit  $\varepsilon \rightarrow 0$  in order to get the desired integral equation

$$\underline{\mathbf{f}}^o(\underline{r}) = \underline{\mathbf{f}}_h^o(\underline{r}) + i\omega \underline{\underline{D}} \cdot \left[ \underline{\mathbf{C}}_c^o - \underline{\mathbf{C}}^o(\underline{r}) \right] \cdot \underline{\mathbf{f}}^o(\underline{r}) + i\omega \text{PV} \int d^3 \underline{r}' \underline{\mathbf{g}}_c^o(\underline{r}, \underline{r}') \cdot \left[ \underline{\mathbf{C}}_c^o - \underline{\mathbf{C}}^o(\underline{r}') \right] \cdot \underline{\mathbf{f}}^o(\underline{r}'), \quad (34)$$

wherein the symbol PV identifies that the integral following it has to be evaluated in the ‘principal-value sense’, as described.

## 4.2 Depolarization Dyadic

The  $6 \times 6$  depolarization dyadic  $\underline{\underline{D}}$  in (34) is a two-dimensional surface integral (Michel & Weiglhofer 1997; Weiglhofer, Lakhtakia, & Michel 1997):

$$\begin{aligned} \underline{\underline{D}} = & -\frac{i\omega}{4\pi k_0^2} \left\{ \int_0^{2\pi} d\phi_q \int_0^\pi d\theta_q \sin \theta_q \right. \\ & \times \left[ \begin{array}{cc} \mu_0 \underline{\underline{U}}^{-1} \cdot \frac{(\hat{\underline{u}}_q \cdot \underline{\underline{U}}^{-1})(\hat{\underline{u}}_q \cdot \underline{\underline{U}}^{-1})}{(\hat{\underline{u}}_q \cdot \underline{\underline{U}}^{-1}) \cdot \underline{\underline{\epsilon}}_c^o \cdot (\hat{\underline{u}}_q \cdot \underline{\underline{U}}^{-1})} \cdot \underline{\underline{U}}^{-1} & \underline{\underline{0}} \\ \underline{\underline{0}} & \epsilon_0 \underline{\underline{U}}^{-1} \cdot \frac{(\hat{\underline{u}}_q \cdot \underline{\underline{U}}^{-1})(\hat{\underline{u}}_q \cdot \underline{\underline{U}}^{-1})}{(\hat{\underline{u}}_q \cdot \underline{\underline{U}}^{-1}) \cdot \underline{\underline{\mu}}_c^o \cdot (\hat{\underline{u}}_q \cdot \underline{\underline{U}}^{-1})} \cdot \underline{\underline{U}}^{-1} \end{array} \right] \Big\}. \quad (35) \end{aligned}$$

In general, it has to be evaluated numerically, which can be accomplished quite easily by using the Gauss–Legendre quadrature scheme, for instance (Mackay & Weiglhofer, 2000).

In two special situations,  $\underline{\underline{\mathbf{D}}}$  can be obtained analytically. Suppose, first, that the ellipsoidal region  $V_\varepsilon$  is spherical. Then,  $\underline{\underline{\mathbf{U}}} = \underline{\underline{\mathbf{I}}}$  so that

$$\underline{\underline{\mathbf{D}}} = -\frac{i\omega}{4\pi k_0^2} \int_0^{2\pi} d\phi_q \int_0^\pi d\theta_q \sin\theta_q \begin{bmatrix} \mu_0 \frac{\hat{\underline{\underline{\mathbf{u}}}_q \cdot \hat{\underline{\underline{\mathbf{u}}}_q}}{\hat{\underline{\underline{\mathbf{u}}}_q \cdot \underline{\underline{\mathbf{e}}}_c^o \cdot \hat{\underline{\underline{\mathbf{u}}}_q}} & \underline{\underline{\mathbf{0}}} \\ \underline{\underline{\mathbf{0}}} & \epsilon_0 \frac{\hat{\underline{\underline{\mathbf{u}}}_q \cdot \hat{\underline{\underline{\mathbf{u}}}_q}}{\hat{\underline{\underline{\mathbf{u}}}_q \cdot \underline{\underline{\mu}}_c^o \cdot \hat{\underline{\underline{\mathbf{u}}}_q}} \end{bmatrix}. \quad (36)$$

For  $\underline{\underline{\mathbf{e}}}_c^o$  defined in (7) and  $\underline{\underline{\mu}}_c^o$  defined in (8), the two integrals on the right side of (36) can be evaluated in closed form to yield (Mackay & Lakhtakia 2005)

$$\begin{aligned} \underline{\underline{\mathbf{D}}} = & -\frac{i\omega}{4\pi k_0^2} \begin{bmatrix} \frac{\mu_0}{1-8a_c} & \underline{\underline{\mathbf{0}}} \\ \underline{\underline{\mathbf{0}}} & \epsilon_0(1-8a_c) \end{bmatrix} \\ & \times \begin{bmatrix} \Gamma_t(\rho^\epsilon) (\underline{\underline{\mathbf{I}}} - \underline{\underline{\mathbf{u}}}_c \underline{\underline{\mathbf{u}}}_c) + \Gamma_c(\rho^\epsilon) \underline{\underline{\mathbf{u}}}_c \underline{\underline{\mathbf{u}}}_c & \underline{\underline{\mathbf{0}}} \\ \underline{\underline{\mathbf{0}}} & \Gamma_t(\rho^\mu) (\underline{\underline{\mathbf{I}}} - \underline{\underline{\mathbf{u}}}_c \underline{\underline{\mathbf{u}}}_c) + \Gamma_c(\rho^\mu) \underline{\underline{\mathbf{u}}}_c \underline{\underline{\mathbf{u}}}_c \end{bmatrix} \end{aligned} \quad (37)$$

where

$$\Gamma_t(\rho) = \begin{cases} \frac{1}{2} \left( \frac{1}{1-\rho} - \frac{\rho \sinh^{-1} \sqrt{\frac{1-\rho}{\rho}}}{(1-\rho)^{\frac{3}{2}}} \right) & \text{for } 0 < \rho < 1 \\ \frac{1}{2} \left( \frac{\rho \sec^{-1} \sqrt{\rho}}{(\rho-1)^{\frac{3}{2}}} - \frac{1}{\rho-1} \right) & \text{for } \rho > 1 \end{cases}, \quad (38)$$

$$\Gamma_c(\rho) = \begin{cases} \frac{\sinh^{-1} \sqrt{\frac{1-\rho}{\rho}}}{(1-\rho)^{\frac{3}{2}}} - \frac{1}{1-\rho} & \text{for } 0 < \rho < 1 \\ \frac{1}{\rho-1} - \frac{\sec^{-1} \sqrt{\rho}}{(\rho-1)^{\frac{3}{2}}} & \text{for } \rho > 1 \end{cases}, \quad (39)$$

$$(40)$$

and

$$\left. \begin{aligned} \rho^\epsilon &= \frac{1+20a_c}{1-8a_c} \\ \rho^\mu &= \frac{1-8a_c}{1-24a_c} \end{aligned} \right\}. \quad (41)$$

The second special situation comes when one of the semi-axes of the ellipsoidal region  $V_\varepsilon$  coincides with  $\underline{\underline{\mathbf{u}}}_c$ . Without loss of generality, let us choose  $\underline{\underline{\mathbf{u}}}_c = \hat{\underline{\underline{\mathbf{u}}}}_z$  and then introduce the  $3 \times 3$

dyadics

$$\left. \begin{aligned} \underline{\underline{\epsilon}}_c^{o'} &= \underline{\underline{U}}^{-1} \cdot \underline{\underline{\epsilon}}_c^o \cdot \underline{\underline{U}}^{-1} = \epsilon_x^{o'} \hat{u}_x \hat{u}_x + \epsilon_y^{o'} \hat{u}_y \hat{u}_y + \epsilon_z^{o'} \hat{u}_z \hat{u}_z \\ \underline{\underline{\mu}}_c^{o'} &= \underline{\underline{U}}^{-1} \cdot \underline{\underline{\mu}}_c^o \cdot \underline{\underline{U}}^{-1} = \mu_x^{o'} \hat{u}_x \hat{u}_x + \mu_y^{o'} \hat{u}_y \hat{u}_y + \mu_z^{o'} \hat{u}_z \hat{u}_z \end{aligned} \right\}, \quad (42)$$

wherein

$$\left. \begin{aligned} \epsilon_x^{o'} &= \frac{1-8a_c}{a_x^2} \\ \epsilon_y^{o'} &= \frac{1-8a_c}{a_y^2} \\ \epsilon_z^{o'} &= \frac{1+20a_c}{a_z^2} \end{aligned} \right\}, \quad \left. \begin{aligned} \mu_x^{o'} &= \frac{1}{(1-8a_c)a_x^2} \\ \mu_y^{o'} &= \frac{1}{(1-8a_c)a_y^2} \\ \mu_z^{o'} &= \frac{1}{(1-24a_c)a_z^2} \end{aligned} \right\}. \quad (43)$$

It then transpires that the depolarization dyadic may be represented as (Weiglhofer, 1998)

$$\underline{\underline{D}} = -\frac{i\omega}{4\pi k_0^2} \begin{bmatrix} \mu_0 \underline{\underline{U}}^{-1} \cdot \underline{\underline{d}}^\epsilon \cdot \underline{\underline{U}}^{-1} & \underline{\underline{0}} \\ \underline{\underline{0}} & \epsilon_0 \underline{\underline{U}}^{-1} \cdot \underline{\underline{d}}^\mu \cdot \underline{\underline{U}}^{-1} \end{bmatrix}, \quad (44)$$

where

$$\underline{\underline{d}}^\eta = d_x^\eta \hat{u}_x \hat{u}_x + d_y^\eta \hat{u}_y \hat{u}_y + d_z^\eta \hat{u}_z \hat{u}_z, \quad (\eta = \epsilon, \mu), \quad (45)$$

with

$$\left. \begin{aligned} d_x^\eta &= \frac{(\eta_y^{o'})^{1/2} [F(\lambda_1, \lambda_2) - E(\lambda_1, \lambda_2)]}{(\eta_y^{o'} - \eta_x^{o'}) (\eta_z^{o'} - \eta_x^{o'})^{1/2}} \\ d_y^\eta &= \frac{1}{\eta_y^{o'} - \eta_x^{o'}} \left\{ \frac{\eta_x^{o'} - \eta_y^{o'}}{\eta_z^{o'} - \eta_y^{o'}} - \left( \frac{\eta_z^{o'} - \eta_x^{o'}}{\eta_y^{o'}} \right)^{1/2} \right. \\ &\quad \times \left[ \frac{\eta_x^{o'}}{\eta_z^{o'} - \eta_x^{o'}} F(\lambda_1, \lambda_2) - \frac{\eta_y^{o'}}{\eta_z^{o'} - \eta_y^{o'}} E(\lambda_1, \lambda_2) \right] \left. \right\} \\ d_z^\eta &= \frac{1}{\eta_z^{o'} - \eta_y^{o'}} \left\{ 1 - \left( \frac{\eta_y^{o'}}{\eta_z^{o'} - \eta_x^{o'}} \right)^{1/2} E(\lambda_1, \lambda_2) \right\} \end{aligned} \right\}, \quad (\eta = \epsilon, \mu), \quad (46)$$

which involve  $F(\lambda_1, \lambda_2)$  and  $E(\lambda_1, \lambda_2)$  as elliptic integrals of the first and second kinds (Gradshteyn & Ryzhik, 1980), respectively, with arguments

$$\left. \begin{aligned} \lambda_1 &= \tan^{-1} \left( \frac{\eta_z^{o'} - \eta_x^{o'}}{\eta_x^{o'}} \right)^{1/2} \\ \lambda_2 &= \left[ \frac{\eta_z^{o'} (\eta_y^{o'} - \eta_x^{o'})}{\eta_y^{o'} (\eta_z^{o'} - \eta_x^{o'})} \right]^{1/2} \end{aligned} \right\}, \quad (\eta = \epsilon, \mu). \quad (47)$$

### 4.3 Numerical–Solution Techniques

As may be easily guessed, (34) shall have to be solved numerically. When the spatial variations of  $\underline{B}^{dc}(\underline{r})$  are confined to some bounded region  $V_i$  whereas the magnetostatic field is uniform with a value  $\underline{B}_c$  everywhere outside  $V_i$ , the method of moments (Miller, Medgyesi–Mitschang, & Newman, 1991; Wang, 1991) and the coupled dipole method (Purcell & Pennypacker, 1973; Lakhtakia, 1990) offer relatively easy algorithms to implement. Both methods are related to each other (Lakhtakia, 1992), and their adaptations for the present purposes are described as follows.

#### 4.3.1 Method of Moments

Implementation of the method of moments requires that  $V_i$  be replaced by a lattice of points  $\underline{r}_n$ ,  $n \in [1, N]$ . Attached to every  $\underline{r}_n$  is an electrically small region described by a shape dyadic  $\underline{\underline{U}}_n$  and of volume  $v_n$ ; the sum  $(\sum_{n=1}^N v_n)$  equals the volume of  $V_i$ . Furthermore, a depolarization dyadic  $\underline{\underline{D}}_n$  is associated with every  $\underline{r}_n$ .

Equation (34) is specialized to  $\underline{r} = \underline{r}_n$ ,  $n \in [1, N]$ , to obtain the set of algebraic equations

$$\begin{aligned} \underline{\mathbf{f}}^o(\underline{r}_n) \approx & \underline{\mathbf{f}}_h^o(\underline{r}_n) + i\omega \underline{\underline{D}}_n \cdot [\underline{\underline{\mathbf{C}}}_c^o - \underline{\underline{\mathbf{C}}}^o(\underline{r}_n)] \cdot \underline{\mathbf{f}}^o(\underline{r}_n) \\ & + \sum_{m=1, m \neq n}^N v_m \underline{\underline{\mathbf{g}}}_c^o(\underline{r}_n, \underline{r}_m) \cdot [\underline{\underline{\mathbf{C}}}_c^o - \underline{\underline{\mathbf{C}}}^o(\underline{r}_m)] \cdot \underline{\mathbf{f}}^o(\underline{r}_m), \quad n \in [1, N], \end{aligned} \quad (48)$$

where  $\underline{\mathbf{f}}_h^o(\underline{r})$  represents the incident optical field (i.e., the optical fields when  $\underline{B}^{dc}(\underline{r}) = \underline{B}_c \forall \underline{r} \in V_i$ ).

This set of equations is rewritten as

$$\underline{\mathbf{f}}^o(\underline{r}_n) = \sum_{m=1}^N \underline{\underline{\mathbf{Q}}}_{nm}^{MOM} \cdot \underline{\mathbf{f}}^o(\underline{r}_m), \quad n \in [1, N], \quad (49)$$

where

$$\underline{\underline{\mathbf{Q}}}_{nm}^{MOM} = \left\{ \underline{\underline{\mathbf{I}}} - i\omega \underline{\underline{D}}_n \cdot [\underline{\underline{\mathbf{C}}}_c^o - \underline{\underline{\mathbf{C}}}^o(\underline{r}_n)] \right\} \delta_{nm} - v_m \underline{\underline{\mathbf{g}}}_c^o(\underline{r}_n, \underline{r}_m) \cdot [\underline{\underline{\mathbf{C}}}_c^o - \underline{\underline{\mathbf{C}}}^o(\underline{r}_m)] (1 - \delta_{nm}), \quad (50)$$

with  $\delta_{nm}$  as the Kronecker delta function.

Equations (49) are solved for all  $\underline{\mathbf{f}}^o(\underline{r}_n)$ ,  $n \in [1, N]$  by a variety of numerical techniques (Carnahan, Luther, & Wilkes, 1969), with the conjugate gradient method being the preferred technique when  $N$  is large (Strang, 1986; Sarkar, 1991). Once all  $\underline{\mathbf{f}}^o(\underline{r}_n)$  are known, the total field at any  $\underline{r} \notin V_i$  may be determined from (34) as

$$\underline{\mathbf{f}}^o(\underline{r}) \approx \underline{\mathbf{f}}_h^o(\underline{r}) + \sum_{m=1}^N v_m \underline{\underline{\mathbf{g}}}_c^o(\underline{r}, \underline{r}_m) \cdot [\underline{\underline{\mathbf{C}}}_c^o - \underline{\underline{\mathbf{C}}}^o(\underline{r}_m)] \cdot \underline{\mathbf{f}}^o(\underline{r}_m), \quad \underline{r} \notin V_i. \quad (51)$$

The difference  $\mathbf{f}^o(\underline{r}) - \mathbf{f}_h^o(\underline{r})$  yields the scattered optical field at  $\underline{r} \notin V_i$ .

### 4.3.2 Coupled Dipole Method

In the coupled dipole method, the first and the third terms on the right side of (48) are used to define an *exciting* optical field via

$$\mathbf{f}_{exc}^o(\underline{r}_n) \approx \mathbf{f}_h^o(\underline{r}_n) + \sum_{m=1, m \neq n}^N v_m \underline{\mathbf{g}}_c^o(\underline{r}_n, \underline{r}_m) \cdot \left[ \underline{\mathbf{C}}_c^o - \underline{\mathbf{C}}^o(\underline{r}_m) \right] \cdot \mathbf{f}^o(\underline{r}_m), \quad n \in [1, N]. \quad (52)$$

Thereafter, the set of equations (48) is rewritten as

$$\mathbf{f}^o(\underline{r}_n) \approx \mathbf{f}_{exc}^o(\underline{r}_n) + i\omega \underline{\mathbf{D}}_n \cdot \left[ \underline{\mathbf{C}}_c^o - \underline{\mathbf{C}}^o(\underline{r}_n) \right] \cdot \mathbf{f}^o(\underline{r}_n), \quad n \in [1, N], \quad (53)$$

whence

$$\mathbf{f}^o(\underline{r}_n) \approx \left\{ \underline{\mathbf{I}} - i\omega \underline{\mathbf{D}}_n \cdot \left[ \underline{\mathbf{C}}_c^o - \underline{\mathbf{C}}^o(\underline{r}_n) \right] \right\}^{-1} \cdot \mathbf{f}_{exc}^o(\underline{r}_n), \quad n \in [1, N]. \quad (54)$$

Clearly,  $\mathbf{f}_{exc}^o(\underline{r}_n)$  contains not only the incident optical field at  $\underline{r}_n$  but also the scattered optical field due to all other lattice points in  $V_i$ .

Substituting (54) in (52), we obtain

$$\mathbf{f}_h^o(\underline{r}_n) = \sum_{m=1}^N \underline{\mathbf{Q}}_{nm}^{CDM} \cdot \mathbf{f}_{exc}^o(\underline{r}_m), \quad n \in [1, N], \quad (55)$$

where

$$\underline{\mathbf{Q}}_{nm}^{CDM} = \underline{\mathbf{I}} \delta_{nm} - i\omega v_m \underline{\mathbf{g}}_c^o(\underline{r}_n, \underline{r}_m) \cdot \underline{\mathbf{X}}_m (1 - \delta_{nm}) \quad (56)$$

contains

$$\underline{\mathbf{X}}_m = \left[ \underline{\mathbf{C}}_c^o - \underline{\mathbf{C}}^o(\underline{r}_m) \right] \cdot \left\{ \underline{\mathbf{I}} - i\omega \underline{\mathbf{D}}_m \cdot \left[ \underline{\mathbf{C}}_c^o - \underline{\mathbf{C}}^o(\underline{r}_m) \right] \right\}^{-1}. \quad (57)$$

Equations (55) have to be solved numerically too — just like (49) — in order to determine  $\mathbf{f}_{exc}^o(\underline{r}_n)$ ,  $n \in [1, N]$ . Thereafter, by virtue of (51) and (54), the total optical field at any  $\underline{r} \notin V_i$  may be determined from (34) as

$$\mathbf{f}^o(\underline{r}) \approx \mathbf{f}_h^o(\underline{r}) + \sum_{m=1}^N v_m \underline{\mathbf{g}}_c^o(\underline{r}, \underline{r}_m) \cdot \underline{\mathbf{X}}_m \cdot \mathbf{f}_{exc}^o(\underline{r}_m), \quad \underline{r} \notin V_i. \quad (58)$$

The reason for the name of this technique is that the product  $v_m \underline{\mathbf{X}}_m$  in (56) and (58) may be considered as a  $6 \times 6$  polarizability dyadic of the electrically small region associated with  $\underline{r}_m$  (Weiglhofer, Lakhtakia, & Michel, 1997; Lakhtakia & Weiglhofer, 2000).

#### 4.4 Homogenization of Magnetostatic Field

The coupled dipole method naturally leads to the homogenization problem of replacing a spatially varying  $\underline{B}^{dc}(\underline{r})$  in some region  $\tilde{V}$  by a uniform magnetostatic field  $\underline{B}_{uni}$  for the analysis of optical fields in  $\tilde{V}$ .

Following the strong-property-fluctuation theory (SPFT) (Tsang & Kong, 1981; Michel & Lakhtakia, 1995; Mackay, Lakhtakia, & Weiglhofer 2000), let us begin by introducing the continuous analog of the exciting optical field of (54) as

$$\underline{\mathbf{f}}_{exc}^o(\underline{r}) = \left\{ \underline{\mathbf{I}} - i\omega \underline{\mathbf{D}} \cdot \left[ \underline{\mathbf{C}}_{uni}^o - \underline{\mathbf{C}}^o(\underline{r}) \right] \right\} \cdot \underline{\mathbf{f}}^o(\underline{r}), \quad \underline{r} \in \tilde{V}, \quad (59)$$

where  $\underline{\mathbf{C}}_{uni}^o$  is defined in analogy with  $\underline{\mathbf{C}}_c^o$  — as per (7), (8), and (13) — and  $\underline{\mathbf{D}}$  is to be calculated by using  $\underline{\mathbf{C}}_c^o = \underline{\mathbf{C}}_{uni}^o$  and  $\underline{\mathbf{U}} = \underline{\mathbf{I}}$  on the right side of (35). Thereby, the integral equation (34) may be rewritten as

$$\underline{\mathbf{f}}_{exc}^o(\underline{r}) = \underline{\mathbf{f}}_h^o(\underline{r}) + i\omega \text{PV} \int_{\tilde{V}} d^3 \underline{r}' \underline{\mathbf{g}}_{uni}^o(\underline{r}, \underline{r}') \cdot \underline{\mathbf{x}}(\underline{r}') \cdot \underline{\mathbf{f}}_{exc}^o(\underline{r}'), \quad (60)$$

where  $\underline{\mathbf{g}}_{uni}^o(\underline{r}, \underline{r}')$  is defined in analogy with  $\underline{\mathbf{g}}_c^o(\underline{r}, \underline{r}')$  in (16) with  $\underline{\mathbf{C}}_c^o = \underline{\mathbf{C}}_{uni}^o$ , and the polarizability density dyadic is defined as

$$\underline{\mathbf{x}}(\underline{r}) = \left[ \underline{\mathbf{C}}_{uni}^o - \underline{\mathbf{C}}^o(\underline{r}) \right] \cdot \left\{ \underline{\mathbf{I}} - i\omega \underline{\mathbf{D}} \cdot \left[ \underline{\mathbf{C}}_{uni}^o - \underline{\mathbf{C}}^o(\underline{r}) \right] \right\}^{-1} \quad (61)$$

in analogy to  $\underline{\mathbf{x}}_m$ .

Within the SPFT framework, the volume integrals of both sides of (60) are used to compute the constitutive parameters of a uniform medium which approximate to those of the medium described by  $\underline{\mathbf{C}}^o(\underline{r})$ . The simplest estimate of these constitutive parameters is provided by  $\underline{\mathbf{C}}_{uni}^o$ , which is found by imposing the condition

$$\int_{\tilde{V}} d^3 \underline{r} \underline{\mathbf{x}}(\underline{r}) = \underline{\mathbf{0}}. \quad (62)$$

In order to deduce  $\underline{B}_{uni}$  from (62), we introduce the piecewise-uniform approximation

$$\underline{B}^{dc}(\underline{r}) = \underline{B}_j, \quad \underline{r} \in \tilde{V}_j, \quad (63)$$

with  $\tilde{V} = \bigcup_{j=1}^J \tilde{V}_j$ . The corresponding uniform  $6 \times 6$  constitutive dyadics are defined as

$$\underline{\mathbf{C}}^o(\underline{r}) = \underline{\mathbf{C}}_j^o = \begin{bmatrix} \epsilon_0 \underline{\mathbf{E}}_j^o & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \mu_0 \underline{\mathbf{H}}_j^o \end{bmatrix}, \quad \underline{r} \in \tilde{V}_j, \quad (64)$$

with  $3 \times 3$  dyadic components

$$\left. \begin{aligned} \underline{\underline{\epsilon}}_j^o &= (1 - 8\epsilon_0 c_0^2 \xi \underline{B}_j \cdot \underline{B}_j) \underline{I} + 28\epsilon_0 c_0^2 \xi \underline{B}_j \underline{B}_j \\ \underline{\underline{\mu}}_j^o &= [(1 - 8\epsilon_0 c_0^2 \xi \underline{B}_j \cdot \underline{B}_j) \underline{I} - 16\epsilon_0 c_0^2 \xi \underline{B}_j \underline{B}_j]^{-1} \end{aligned} \right\}. \quad (65)$$

Then, the condition (62) implies that

$$\sum_{j=1}^J f_j \left( \underline{\underline{\mathbf{C}}}_{uni}^o - \underline{\underline{\mathbf{C}}}_j^o \right) \cdot \left[ \underline{I} - i\omega \underline{D} \cdot \left( \underline{\underline{\mathbf{C}}}_{uni}^o - \underline{\underline{\mathbf{C}}}_j^o \right) \right]^{-1} = \underline{0}, \quad (66)$$

where  $f_j$  is the volumetric proportion of region  $\tilde{V}_j$  relative to  $\tilde{V}$ . This nonlinear dyadic equation can be straightforwardly solved for  $\underline{\underline{\mathbf{C}}}_{uni}^o$  — and thereby  $\underline{B}_{uni}$  — by standard numerical procedures (Michel, Lakhtakia, & Weiglhofer, 1998).

Higher-order corrections to  $\underline{\underline{\mathbf{C}}}_{uni}^o$  can also be derived using the SPFT, depending upon the statistical details of the spatial fluctuations in  $\underline{B}^{dc}(\underline{r})$  (Mackay, Lakhtakia, & Weiglhofer, 2000).

## 5 Concluding Remarks

To conclude, we have formulated an integral equation for the scattering of light (and other high-frequency) electromagnetic radiation in vacuum by either a strong magnetostatic field or a strong low-frequency magnetic field. This integral equation, set in the language of 6-vectors and  $6 \times 6$  dyadics, exploits the frequency-domain dyadic Green function for a linear, homogeneous, uniaxial, dielectric-magnetic medium. The derived integral equation can be solved numerically, using the method of moments as well as the coupled dipole method. Finally, we have shown that the strong-property-fluctuation theory allows the homogenization of a spatially varying magnetostatic field in the present context.

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## References

Adler, S.L. 1971. Photon splitting and photon dispersion in a strong magnetic field. *Ann. Phys. (NY)* 67:599–647.

- Adler, S.L. 2007. Vacuum birefringence in a rotating magnetic field. *J. Phys. A: Math. Theor.* 40:F143–F152.
- Carnahan, B., H.A. Luther, and J.O. Wilkes. 1969. *Applied Numerical Methods*. New York: Wiley.
- Gradshteyn, I.S., and I.M. Ryzhik. 1980. *Table of Integrals, Series, and Products*. New York: Academic Press.
- Jackson, J.D. 1998. *Classical Electrodynamics, 3rd edition*. New York: Wiley (Section 1.3).
- Lakhtakia, A. 1990. Macroscopic theory of the coupled dipole approximation method. *Opt. Commun.* 79:1–5.
- Lakhtakia, A. 1992. Strong and weak forms of the method of moments and the coupled dipole method for scattering of time-harmonic electromagnetic fields. *Int. J. Modern Phys. C* 3:583–603; corrections: 1993, 4:721–722.
- Lakhtakia, A., and W.S. Weiglhofer. 2000. Maxwell Garnett formalism for weakly nonlinear, bianisotropic, dilute, particulate composite media. *Int. J. Electron.* 87:1401–1408.
- Lakhtakia, A., V.K. Varadan, and V.V. Varadan. 1991. Plane waves and canonical sources in a gyroelectromagnetic uniaxial medium. *Int. J. Electron.* 71:853–861.
- Mackay, T.G., and A. Lakhtakia. 2005. Anisotropic enhancement of group velocity in a homogenized dielectric composite medium. *J. Opt. A: Pure Appl. Opt.* 7:669–674.
- Mackay, T.G., and W. S. Weiglhofer. 2000. Homogenization of biaxial composite materials: dissipative anisotropic properties. *J. Opt. A: Pure Appl. Opt.* 2:426–432.
- Mackay, T.G., A. Lakhtakia, and W.S. Weiglhofer. 2001. Strong-property-fluctuation theory for homogenization of bianisotropic composites: formulation. *Phys. Rev. E* 62:6052–6064; corrections: 2001, 63:049901.
- Michel, B., and A. Lakhtakia. 1995. Strong-property-fluctuation theory for homogenizing chiral particulate composites. *Phys. Rev. E* 51:5701–5707.
- Michel, B., and W. S. Weiglhofer. 1997. Pointwise singularity of dyadic Green function in a general bianisotropic medium. *Arch. Elektron. Übertrag.* 51:219–223; correction: 1998, 52:31.
- Michel, B., A. Lakhtakia, and W.S. Weiglhofer. 1998. Homogenization of linear bianisotropic particulate composite media — Numerical studies. *Int. J. Appl. Electromag. Mech.* 9:167–178; corrections: 1999, 10:537–538.
- Miller, E.K., L. Medgyesi-Mitschang, and E. H. Newman. 1991. *Computational Electromagnet-*

- ics: Frequency-Domain Method of Moments*. New York: IEEE Press.
- Purcell, E.M., and C.R. Pennypacker. 1973. Scattering and absorption of light by nonspherical dielectric grains. *Astrophys. J.* 186:705–714.
- Sarkar, T.K. (ed). 1991. *Application of Conjugate Gradient Method to Electromagnetics and Signal Analysis*. New York: Elsevier.
- Singh, O.N., and A. Lakhtakia (eds). 2000. *Electromagnetic Fields in Unconventional Materials and Structures*. New York: Wiley.
- Strang, G. 1986. *Introduction to Applied Mathematics*. Wellesley, MA: Wellesley–Cambridge Press.
- Tsang, L., and J.A. Kong. 1981. Scattering of electromagnetic waves from random media with strong permittivity fluctuations. *Radio Sci.* 16:303–320.
- Wang, J.J.H. 1991. *Generalized Moment Methods in Electromagnetics*. New York: Wiley.
- Weiglhofer, W.S. 1990. Dyadic Green’s functions for general uniaxial media. *IEE Proc., Pt. H* 137:5–10.
- Weiglhofer, W.S. 1998. Electromagnetic depolarization dyadics and elliptic integrals. *J. Phys. A: Math. Gen.* 31:7191–7196.
- Weiglhofer, W.S., and A. Lakhtakia (eds). 2003. *Introduction to Complex Mediums for Optics and Electromagnetics*. Bellingham, WA, USA: SPIE Press.
- Weiglhofer, W.S., A. Lakhtakia, and B. Michel. 1997. Maxwell Garnett and Bruggeman formalism for a particulate composite with bianisotropic host medium. *Microw. Opt. Technol. Lett.* 15:263–266; correction: 1999, 22:221.